11p.

X63-11418
code 2d

NASA TT F-8208

ON COMPLETE AVERAGING WITH SEVERAL INTERMEDIATE ELEMENTS IN THE CANONICAL PROBLEM OF CELESTIAL MECHANICS

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WASHINGTON April 1962

NASA TT F-8208

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[Following is the translation of an article by N. D. Moiseyev in Trudy Gosudarstvennogo Astronomicheskogo Instituta im. P. K. Shernberga (Transactions of the P. K. Sternberg State Astronomical Institute), Vol. 24, pp. 10 - 16, (1954).]

Introduction

The present article is one contribution in our series of works devoted to the consideration of simplified variants on the problem of celestial mechanics which are obtained by averaging. More exactly, the present article relates to that portion of our work which treats of the so-called "interpolatively averaged" variants of the problem of celestial mechanics.

Section 1. Complete Averaging of the Canonical Problem with Several Intermediate Elements

Let a canonical problem be given which has the following equations of motion:

$$\frac{d\xi_k}{dt} = \frac{\partial H}{\partial \eta_k}; \quad \frac{d\eta_k}{dt} = -\frac{\partial H}{\partial \xi_k}; \quad (k = 1, 2, \ldots n), \tag{1}$$

$$H = H(\xi_1; \ldots \xi_n; \eta_1 \ldots \eta_n; t). \tag{2}$$

In particular cases the time t may or may not appear in the Hamiltonian (2).

In addition, let there be defined a set of m functions ζ_l of the canonical variables (or canonical elements) of the given problem:

$$\zeta_l = \zeta_l(\xi_1; \ldots \xi_n; \eta_1; \ldots \eta_n); \quad (l = 1, 2, \ldots m). \tag{3}$$

We shall hereafter call these quantities ζ_l "intermediate elements".

We arrange the entire set of 2n canonical elements

$$(\xi_1; \xi_2; \ldots \xi_n; \eta_1; \eta_2; \ldots \eta_n) \tag{4}$$

into two groups. One of these, which is composed of m arbitrary canonical elements,

$$(\xi_1; \xi_2 \ldots \xi_{m'}; \eta_1; \eta_2 \ldots \eta_{m''}); \quad m' + m'' = m, \tag{5}$$

we shall call "the basic group". The second group, which consists of the 2n - m remaining canonical elements,

$$(\xi_{m'+1}; \ldots \zeta_n; \eta_{m''+1}; \ldots \eta_n), \tag{6}$$

we shall call "the complementary group".

We solve equations (3) for the "basic" canonical elements, obtaining the formulas

$$\left.\begin{aligned} \xi_{k'} &= \xi_{k'}(\zeta_1;\ldots\zeta_m;\xi_{m'+1};\ldots\xi_n;\eta_{m''+1};\ldots\eta_n);\ (k'=1,2\ldots m'),\\ \eta_{k''} &= \eta_{k''}(\zeta_1;\ldots\zeta_m;\xi_{m'+1};\ldots\xi_n;\eta_{m''+1};\ldots\eta_n);\ (k''=1,2,\ldots m''). \end{aligned}\right\} \quad (7)$$

We then eliminate the "basic" canonical elements from the Hamiltonian (2) by means of these formulas. We will designate the result of this elimination by the symbol

$$H^* = H^*(\zeta_1;\ldots\zeta_m;\xi_{m'+1};\ldots\xi_n;\eta_{m''+1};\ldots\eta_n;t). \quad (8)$$

We average this transformed Hamiltonian, H*, over any given ranges

$$\left.\begin{aligned} \underline{\xi}_{k'} \leqslant \xi_{k'} \leqslant \overline{\xi}_{k'};\ (k'=m'+1;\ldots n),\\ \underline{\eta}_{k''} \leqslant \eta_{k''} \leqslant \overline{\eta}_{k''};\ (k''=m''+1;\ldots n). \end{aligned}\right\} \quad (9)$$

of the "complementary" canonical elements, (6).

We effect the above aver-

$$\widetilde{H} = \frac{1}{\prod\limits_{k'=m'+1}^{n}(\overline{\xi}_{k'}-\underline{\xi}_{k'})\prod\limits_{k''=m''+1}^{n}(\overline{\eta}_{k''}-\underline{\eta}_{k''})}\int\limits_{\xi_{m'+1}=\underline{\xi}_{m'+1}}^{\overline{\xi}_{m'+1}}\cdots\int\limits_{\eta_n=\underline{\eta}_n}^{\overline{\eta}_n} H^*\,d\xi_{m'+1}\ldots d\eta_n \quad (10)$$

while observing the following three requirements:

1. The intermediate elements (3) are considered invariant during the integration.

2. The quantity t is considered invariant during the integration.

3. The "complementary" canonical variables (6), with respect to which the average is taken, are regarded as mutually independent variables during the integration.

We will describe the above averaged Hamiltonian as completely averaged with m intermediate elements (3).

The "completely averaged" Hamiltonian, $\widetilde{H}$, so obtained will clearly depend only on the m intermediate elements (3) and the time t,

$$\widetilde{H}=\widetilde{H}(\zeta_1;\ldots\zeta_m;\ t). \qquad (11)$$

We will henceforth regard the intermediate elements (3) in this formula actually as intermediate variables and consider that the completely averaged Hamiltonian, $\widetilde{H}$, depends on all the canonical elements (4) through the intermediate elements (3):

$$\widetilde{H}=\widetilde{H}[\zeta_1(\xi_1;\ldots\xi_n;\ \eta_1;\ldots\eta_n);\ldots\zeta_m(\xi_1;\ldots\eta_n;\ t);\ t]. \qquad (12)$$

Let us now form the canonical differential equations which are obtained from the given equations (1) by replacing the true Hamiltonian, H, by the result, $\widetilde{H}$, of its complete averaging. We shall obtain

$$\frac{d\xi_k}{dt}=\frac{\partial\widetilde{H}}{\partial\eta_k};\qquad \frac{d\eta_k}{dt}=-\frac{\partial\widetilde{H}}{\partial\xi_k};\qquad (k=1,\ 2,\ldots n). \qquad (13)$$

The problem described by this set of equations (13) we will simply call the variant of the given problem obtained by complete averaging with the m intermediate elements (3).

Equations (13) may also be written in the following modified form:

$$\frac{d\xi_k}{dt}=\sum_{l=1}^{m}\frac{\partial\widetilde{H}}{\partial\zeta_l}\,\frac{\partial\zeta_l}{\partial\eta_k};\qquad \frac{d\eta_k}{dt}=-\sum_{l=1}^{m}\frac{\partial\widetilde{H}}{\partial\zeta_l}\,\frac{\partial\zeta_l}{\partial\xi_k}; \qquad (14)$$

$$(k=1,\ 2,\ldots n).$$

Section 2. Some Special Cases of Intermediate Elements and Hamiltonians

It is of interest to consider certain special cases of intermediate elements. Deserving our primary attention among these are the cases in which formulas (3) have the special form

$$\zeta_l=\sum_{k=1}^{n}(r_{lk}\xi_k+s_{lk}\eta_k);\qquad (l=1,\ 2,\ldots m), \qquad (15)$$

where

$$r_{lk};\quad s_{lk} \qquad (16)$$

are given numbers.

In this case of linear intermediate elements, the equations of the completely averaged problem (14) take on the simple form

$$\frac{d\xi_k}{dt}=\sum_{l=1}^{m}\frac{\partial\widetilde{H}}{\partial\zeta_l}s_{lk};\quad \frac{d\eta_k}{dt}=-\sum_{l=1}^{m}\frac{\partial\widetilde{H}}{\partial\zeta_l}r_{lk};\quad (k=1,\ 2,\ \ldots n). \tag{17}$$

Among more general special cases of intermediate elements the following merit attention also:

$$\zeta_l=\sum_{k=1}^{n}[f_{lk}(\xi_k)+g_{lk}(\eta_k)];\qquad (l=1,\ 2,\ldots m), \tag{18}$$

where

$$f_{lk}(\xi_k);\qquad g_{lk}(\eta_k) \tag{19}$$

are given functions of the indicated variables, which, for example, may be algebraic polynomials, etc.

It is of interest to consider the following special cases of completely averaged Hamiltonians $\widetilde{H}$ with respect to their dependence on the intermediate elements, (3).

a) First among these special cases is that in which $\widetilde{H}$ takes the form

$$\widetilde{H}=\widetilde{H}\left(\widetilde{\widetilde{H}};t\right), \tag{20}$$

where the auxiliary function $\widetilde{\widetilde{H}}$ does not depend on the time t,

$$\widetilde{\widetilde{H}}=\widetilde{\widetilde{H}}(\zeta_1;\ldots\zeta_m). \tag{21}$$

b) A second interesting case is that in which $\widetilde{H}$ is defined by the formula

$$\widetilde{H}=\sum_{l=1}^{m}\widetilde{H}_l, \tag{22}$$

where each of the component functions of $\widetilde{H}$ depends on only one intermediate element with corresponding subscript, and perhaps also on t,

$$\widetilde{H}_l = \widetilde{H}_l(\zeta_l; t). \tag{23}$$

Some even simpler variants on cases a) and b) are respectively,

a') in which

$$\widetilde{H} = \widetilde{H}(\zeta_1; \ldots \zeta_m) \tag{24}$$

and

b') in which
$$\widetilde{H} = \sum_{l=1}^{m} \widetilde{H}_l(\zeta_l). \tag{25}$$

Section 3. The Differential Equations for the Intermediate Elements and their Integration in Special Cases

On calculating the total time derivatives of the intermediate elements (3) by means of the differential equations (14) of the completely averaged problem, we obtain the following set of differential equations for the intermediate elements:

$$\frac{d\zeta_l}{dt} = \sum_{k=1}^{n} \sum_{\lambda=1}^{m} \frac{\partial \widetilde{H}}{\partial \zeta_\lambda} \left[\frac{\partial \zeta_l}{\partial \xi_k} \frac{\partial \zeta_\lambda}{\partial \eta_k} - \frac{\partial \zeta_l}{\partial \eta_k} \frac{\partial \zeta_\lambda}{\partial \xi_k} \right]; \qquad (l = 1, \ldots m). \tag{26}$$

We shall examine several special cases in which integrals for equations (26) may be found.

First among these cases is that in which there is only one intermediate element, that is, in which m = 1. For this single intermediate element ζ_1 we have from equations (26),

$$\frac{d\zeta_1}{dt} = 0. \tag{27}$$

From this we obtain the integral

$$\zeta_1(\xi_1; \ldots \xi_n; \eta_1 \ldots \eta_n) = \zeta_{10} = \text{const.} \tag{28}$$

In the frequent case where the completely averaged Hamiltonian, $\widetilde{H}$, does not depend on the time, t, that is in case (24), this integral (28) becomes equivalent to the energy integral,

$$\widetilde{H}(\zeta_1) = \text{const.} \tag{29}$$

Let us now consider the case of an arbitrary number of intermediate elements and let us assume that the completely averaged Hamiltonian is expressed by a formula of type (20). We will then have for the auxiliary function $\widetilde{\widetilde{H}}$ the equations

$$\frac{d\widetilde{\widetilde{H}}}{dt} = \frac{\partial \widetilde{H}}{\partial \widetilde{\widetilde{H}}} \sum_{k=1}^{n} \sum_{l=1}^{m} \sum_{\lambda=1}^{m} \frac{\partial \widetilde{\widetilde{H}}}{\partial \zeta_l} \frac{\partial \widetilde{\widetilde{H}}}{\partial \zeta_\lambda} \left[\frac{\partial \zeta_l}{\partial \xi_k} \frac{\partial \zeta_\lambda}{\partial \eta_k} - \frac{\partial \zeta_l}{\partial \eta_k} \frac{\partial \zeta_\lambda}{\partial \xi_k} \right] = 0, \tag{30}$$

from which we obtain the integral

$$\widetilde{\widetilde{H}}(\zeta_1; \ldots \zeta_m) = \text{const.} \tag{31}$$

In case the completely averaged Hamiltonian is independent of time, that is in case (24), this integral (31) coincides with the energy integral for the completely averaged problem.

Let us now consider the special case of linear intermediate elements (15). In this case the differential equations (26) for the intermediate elements take on the form

$$\frac{d\zeta_l}{dt} = \sum_{\lambda=1}^{m} R_{l\lambda} \frac{\partial H}{\partial \zeta_\lambda}; \quad (l = 1, 2, \ldots m), \tag{32}$$

where

$$R_{l\lambda} = \sum_{k=1}^{n} (r_{lk} s_{\lambda k} - s_{lk} r_{\lambda k}) \tag{33}$$

are constants

Equations (32) will be linear in the intermediate elements if the completely averaged Hamiltonian, $\widetilde{H}$, is quadratic in these same elements.

Section 4. Integration of the Differential Equations for the Canonical Elements in Some Special Cases

The set of differential equations (14) for the canonical elements of the completely averaged problem may be completely or partially integrated in closed form in certain cases.

Such is the case, for example, on completely averaging with one intermediate element linear in the canonical variables. Thus when m 1 and formula (15) holds, equations (14) take the form

$$\left.\begin{array}{c} \dfrac{d\xi_k}{dt}=\dfrac{\partial\widetilde{H}(\zeta_1;t)}{\partial\zeta_1}s_{1k}; \qquad \dfrac{d\eta_k}{dt}=-\dfrac{\partial\widetilde{H}(\zeta_1;t)}{\partial\zeta_1}r_{1k}; \\ (k=1,\ 2,\ldots n). \end{array}\right\} \tag{34}$$

In view of integral (28) the right sides of these equations are known functions of the time, whence the integration of the system may be easily completed.

In the somewhat more complex case, $m = 1$ and intermediate element defined by formula (18), equations (14) take the form

$$\left.\begin{array}{c} \dfrac{d\xi_k}{dt}=\dfrac{\partial\widetilde{H}(\zeta_1;t)}{\partial\zeta_1}g_{1k}'(\eta_k); \qquad \dfrac{d\eta_k}{dt}=-\dfrac{\partial\widetilde{H}(\zeta_1;t)}{\partial\zeta_1}f_{1k}'(\xi_k); \\ (k=1,\ 2,\ldots n). \end{array}\right\} \tag{35}$$

From these one derives the following set of n equations in total differentials:

$$f_{1k}'(\xi_k)\,d\xi_k+g_{1k}'(\eta_k)\,d\eta_k=0; \quad (k=1,2,\ldots n), \tag{36}$$

which give the following n first integrals of the problem:

$$f_1(\xi_k)+g_{1k}(\eta_k)=C_k; \qquad (k=1,\ 2,\ldots n). \tag{37}$$

These allow integration of equations (35) by quadratures.

We now turn to the case in which there is more than one intermediate element. If these elements are linear and defined by formulas

(15), the canonical elements will have differential equations of type (17). Assuming that the number of intermediate elements, m, is less than the total number, 2n, of canonical variables, we may by eliminating the m quantities

write (2n - m) mutually independent linear homogeneous equations with constant coefficients for the time derivatives of the canonical elements. Their integration gives the following set of (2n - m) first integrals of the problem which are linear in the canonical elements:

$$\begin{vmatrix} \xi_1; & -s_{11};\dots; & -s_{m1} \\ \cdots & \cdots & \cdots \\ \xi_{m'}; & -s_{1m'};\dots; & -s_{mm'} \\ \eta_1; & r_{11};\dots; & r_{m1} \\ \cdots & \cdots & \cdots \\ \eta_{m''}; & r_{1m''};\dots; & r_{mm''} \\ \xi_{k'}; & -s_{1k'};\dots; & -s_{mk'} \end{vmatrix} = C_{k'}; \quad (k' = m'+1,\dots n), \qquad (39')$$

$$\begin{vmatrix} \xi_1; & -s_{11};\dots; & -s_{m1} \\ \cdots & \cdots & \cdots \\ \xi_{m'}; & -s_{1m'};\dots; & -s_{mm'} \\ \eta_1; & r_{11};\dots & r_{m1} \\ \cdots & \cdots & \cdots \\ \eta_{m''}; & r_{1m''};\dots; & r_{mm''} \\ \eta_{k''}; & r_{1k''};\dots; & r_{mk''} \end{vmatrix} = C_{k''}; \quad (k'' = m''+1,\dots n), \qquad (39'')$$

with $m' + m'' = m$.

These integrals may be rewritten in the form

$$\left.\begin{array}{c} \sum_{q=1}^{m'} R_{q1}(k')\xi_q + \sum_{q=m'+1}^{m} R_{q1}(k')\eta_q + R_{m+1,1}(k')\xi_{k'} = C_{k'}; \\ (k' = m'+1; \;\dots\; n) \end{array}\right\} \qquad (40')$$

and

$$\left.\begin{array}{l}\sum_{q=1}^{m'} S_{q1}(k'')\xi_q + \sum_{q=m'+1}^{m} S_{q1}(k'')\eta_q + S_{m+1,1}(k'')\eta_{k''} = C_{k''}; \\ (k'' = m'' + 1; \ldots n),\end{array}\right\} \qquad (40'')$$

where

$$R_{q1}(k');\ S_{q1}(k'') \qquad (41)$$

are calculable constants, since $C_{k'}$ and $C_{k''}$ are arbitrary constants.

The knowledge of (2n - m) such integrals permits one to decrease the order of the system of equations for the canonical elements from 2n to m. In case the completely averaged Hamiltonian is independent of time, we may complete the integration of the canonical equations by quadratures whenever the number (2n - m) of integrals (40) is not less than n, that is whenever $m \leq n$.

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